FROM HEEGARD SPLITTINGS TO THE HEEGAARD FLOER HOMOLOGY

LUIS MODES

Abstract. We attempt to give an elementary introduction to the Heegaard Floer homology, an important collection of invariants in low-dimensional topology. The main focus is to develop rigorously the theory of Heegaard splittings, and we later shift to a more intuitive approach in order to explain the more advanced topics required to define $\widehat{HF}$.

1. Introduction

Our final goal is to define the Heegaard Floer homology starting from Heegaard splittings, to which we devote a substantial amount of attention. The structure of our topics is inspired by **[1]** and **[6]**. Getting the Heegaard Floer homology of a 3-manifold Y from scratch is a complicated process, which we can briefly summarize it as follows:

$$Y \rightsquigarrow (\Sigma_g, \alpha, \beta, z) \rightsquigarrow (\widehat{CF}(\mathcal{H}), \partial) \rightsquigarrow \widehat{HF}(Y) \tag{1.1}$$

Let us explain what each of these steps is. First, we must associate a Heegaard diagram $(\Sigma_g, \alpha, \beta, z)$ to Y. This diagram, which consists on a surface Σ_g with embedded set of circles $\alpha = (\alpha_1, \dots, \alpha_g)$ and $\beta = (\beta_1, \dots, \beta_g)$ stores all the information of our manifold Y, i.e., we can recover Y from $(\Sigma_g, \alpha, \beta, z)$. Here, z is just a basepoint required for techincal purposes. This is explained in sections 2 and 3. In section 2, we develop the theory of Heegaard splittings, a way to partition 3-manifolds into elementary pieces. Our inspiration for this section is **[2]**. The main result is the following theorem, which will be key in showing that $\widehat{HF}$ is an invariant:

Theorem 1.1. *Any two Heegaard splittings of a closed orientable manifold Y are stably equivalent.*

In essence, this means that our invariant does not depend on the Heegaard splitting we choose. In section 3, we show how to get the Heegaard diagram from the Heegaard splittings. This completes the step $Y \rightsquigarrow (\Sigma_g, \alpha, \beta, z)$. Our next step is to construct a chain complex, a familiar structure from algebraic topology, from this. This involves some advanced notions. First, we associate a configuration space to the Heegaard diagram $\mathcal{H} = (\Sigma_g, \alpha, \beta, z)$. Namely, we consider $\mathrm{Sym}^g(\Sigma_g) = \Sigma_g \times \cdots \times \Sigma_g / \mathfrak{S}_g$, which is the space of g-tuples of points in Σ_g in which the order does not matter. $\mathrm{Sym}^g(\Sigma_g)$ turns out to be a smooth manifold. This product of g copies of Σ_g induces two real tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ embedded in $\mathrm{Sym}^g(\Sigma_g)$. This will be discussed in section 4. Now, these tori will play an important role in constructing the chain complex. Indeed, we define $\widehat{CF}(\mathcal{H})$ as the abelian group generated by the points of the subspace $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Now, to make $\widehat{CF}(\mathcal{H})$ into a chain complex, we must define a map $\partial : \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H})$ with $\partial^2 = 0$. We outline these definitions and construct ∂ in section 5. This shows the step $(\Sigma_g, \alpha, \beta, z) \rightsquigarrow (\widehat{CF}(\mathcal{H}), \partial)$. Finally, after having defined all the required structures, we formalize the last step of equation (1.1) and define $\widehat{HF}(Y)$, one of the variants of the Heegaard Floer homology, completing the step $(\widehat{CF}(\mathcal{H}), \partial) \to \widehat{HF}(Y)$. This is what we are going to cover.

Now, even though we are not going to cover this, it turns out that there are other versions of the Heegaard Floer homology, namely HF^+, HF^-, and HF^∞. There is also a way to relate these invariants to knots and links, and under certain circumstances we can use the Alexander polynomial of the link to help us compute the invariant. An example of this is shown in **[4]**. In general, there are a lot of interesting applications of the Heegaard Floer homology, so we can only exhort the reader to explore them at their own time and interest.

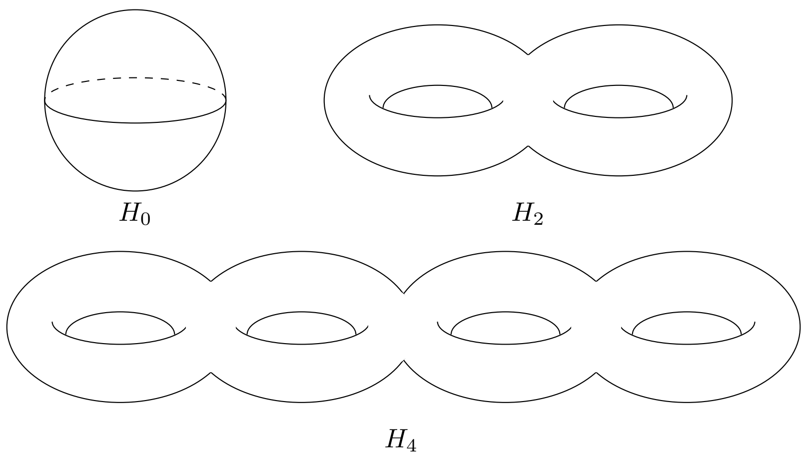
2. Heegaard Splittings

First, let us introduce Heegaard splittings. Intuitively, a Heegard splitting is a way to decompose any 3-manifold into two elementary pieces. We call these pieces handlebodies.

Definition 2.1. A *handlebody* H is an orientable 3-manifold with boundary obtained from B^3 by attaching g copies of $D^2 \times [-1, 1]$ to B^3. The gluing homeomorphisms match the $2g$ discs $D^2 \times \{\pm 1\}$ with $2g$ disjoint discs on ∂B^3 so that the resulting manifold is orientable.

One way to visualize a handlebody is as something diffeomorphic to a neighborhood of $\bigvee_{i=1}^{g} S^1$ in $\mathbb{R}^3$, with the ball B^3 in the center. If we stretch B^3 horizontally, the picture starts to look more like a neighborhood of $\#_{i=1}^{g} S^1$. That is, like a solid version of the genus g surface Σ_g.

Example 2.2. The handlebodies H_0, H_2 and H_4 with 0, 2, and 4 handles, respectively.

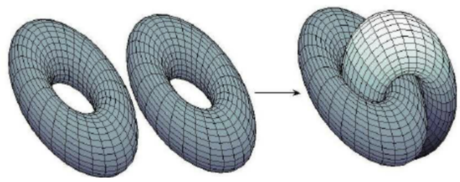


Now, let us define what is a Heegaard splitting.

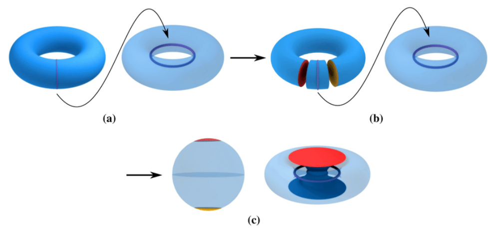
Definition 2.3. If Y is a 3-manifold, a *Heegaard splitting of* Y is a decomposition of the form $Y = H \cup_\varphi H'$, where H, H' are two handlebodies of the same genus, $H \cap H' = \partial H = \partial H'$, and φ is a homeomorphism.

In other words, a Heegaard splitting is decomposing a 3-manifold as two handlebodies glued by their boundary. Even though the definition is simple, it is already hard to visualize, as most 3-manifolds actually live in $\mathbb{R}^4$, just as S^1 lives in $\mathbb{R}^2$. However, let us try to visualize some examples of Heegaard splittings. The first example is $S^3 = B^3 \cup_\varphi B^3$, where φ is just the identity for the boundaries of the two balls. Even though we cannot possibly see this in 3 dimensions, we can get an intuition for it looking at lower dimensions. Note that $S^1 = B^1 \cup B^1$ and $S^2 = B^2 \cup B^2$, where the gluing is also the identity for the boundaries. By the same logic, this holds for $S^3 = B^3 \cup B^3$ (and for S^n in general). As you may have noted, we are slowly dropping the gluing homeomorphism from the notation. Let us describe other two examples.

Example 2.4. We have $S^3 = H_1 \cup_\varphi H_1$, where φ glues the meridian curve on the first torus boundary to the longitude curve on the second torus boundary. However, we also have $S^1 \times S^2 = H_1 \cup_{\varphi'} H_1$, where φ' glues the meridian curve on the first torus with the meridian curve on the second torus and the longitude curve on the first torus with the longitude curve on the second torus.



$S^3 = H_1 \cup_\varphi H_1$



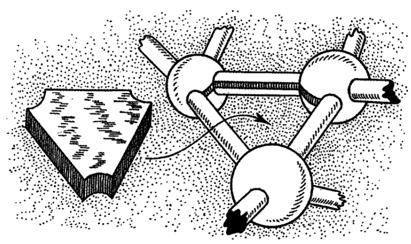
$S^1 \times S^2 = H_1 \cup_{\varphi'} H_1$

Remark 2.5. As we can see from the example, $H \cup_\varphi H'$ and $H \cup_{\varphi'} H'$ can induce different manifolds, even if we use the same handlebodies H, H'.

Now, as we have had a bit us trouble trying to visualize simple examples, we may be tempted to believe that Heegaard splittings are only possible for particular 3-manifolds. However, surprisingly, almost every 3-manifold admits a Heegaard splitting. Furthermore, the proof is beautifully simple.

Theorem 2.6. *Every closed orientable manifold Y admits a Heegaard splitting.*

Proof of Theorem. As Y is a closed manifold, it admits a triangulation. As it is a 3-manifold, its triangulation will actually consists of tetrahedra. Now, thicken the vertices of the triangulation into spheres, the edges into cylinders, and the faces into plates. Note that the interior of the tetrahedra and the plates form a handlebody, and the balls and cylinders form a handlebody of the same genus. Furthermore, Y is closed, so it does not have a boundary. That is, each point in the surface of each tetrahedrum touches another tetrahedrum, so the boundaries of our newly generated handlebodies touch each other everywhere. Hence, these handlebodies of the same genus are glued around their boundaries, so this is a valid Heegaard splitting for Y, as desired. □



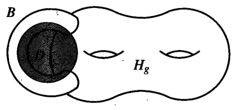
Let us call a Heegaard splitting obtained from a triangulation as in the proof above a Heegaard splitting *associated with a triangulation*. This is good, as it means that we can always consider a Heegaard splitting. Now, we may wonder if Heegaard splittings are unique. The answer is no. In fact, given any triangulation as in the proof of the theorem above, we could make the triangulation finer, and this would yield another valid Heegaard splitting of distinct genus. However, Heegaard splittings are indeed unique modulo an operation called stabilization.

Proposition 2.7. *Given a Heegaard splitting of genus g, it is possible to obtain a Heegaard splitting of genus $g+1$. We call this operation* stabilization.

Proof of Proposition. Consider a Heegaard splitting $Y = H \cup H'$ where H, H' have genus g. Now, add a handle M to H. This new handle and H bound a disc, which we thicken and call N. Now, note that $H \cup M$ and $H' \cup N$ are genus $g+1$ handlebodies. Furthermore, as $M \cup N \cong B^3$, we have that

$$(H \cup M) \cup (H' \cup H) \cong (H \cup H') \cup (M \cup N) \cong Y \cup B^3 \cong Y \tag{2.1}$$

Thus, we have constructed a Heegaard splitting of genus $g+1$. □



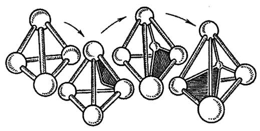
Definition 2.8. Two Heegaard splittings are called *equivalent* if there exists a homeomorphism of M onto itself taking one splitting into the other, and *stably equivalent* if they are equivalent after applying stabilization to each of them a certain number of times.

Finally, let us prove that Heegaard Splittings are indeed unique modulo stabilizations.

Theorem 2.9. *Any two Heegaard splittings of a closed orientable manifold M are stably equivalent.*

Proof of Theorem. We will divide our proof in two claims.

First, let us prove that any two Heegaard splittings associated to triangulations are stably equivalent. Consider two triangulations T and T'. It is known that any two triangulations T and T' have a common subdivision T''. Further, we can note that the Heegaard splitting associated to a subdivision can be obtained from the Heegaard splitting of the original triangulation via stabilization. Thus, the Heegaard splittings associated to T and T' are stably equivalent to the Heegaard splitting associated to T''. Thus, they are stably equivalent, so any two Heegaard splittings associated to a triangulation are stably equivalent.



Now, let us prove that any triangulation is stably equivalent to a Heegaard splitting associated to a triangulation. Let $Y = H \cup H'$ be an arbitrary Heegaard splitting for Y. Let us define some terms. If K is a one-dimensional subcomplex of T, let $U(K)$ denote the space obtained by thickening K. As we saw earlier, $U(K)$ is a handlebody. Now, let $\Gamma \subseteq H$ be the wedge of circles $\bigvee_{i=1}^{g} S^1$ to which H deformation retracts. We claim that there exists a triangulation S of H such that Γ is a subcomplex of H and the following conditions are satisfied:

- The handlebody $U(S^{(1)})$ is obtained from Γ by adding handles, where $S^{(1)}$ denotes the 1-skeleton of S.
- The handlebody $U(S^{(1)})$ is obtained from $U(\partial S^{(1)})$ by adding handles, where $\partial S^{(1)}$ denotes the 1-skeleton of the restriction of S to ∂H.

Most triangulations S satisfy this condition. For example, regard H as a 2-disk with g holes, and take the product triangulation. Note that the conditions above are preserved when taking a finer triangulation. Let $Y = H \cup H'$ be any Heegaard splitting. Choose a triangulation T such that both H and H' are subcomplexes of T. Let S and S' be the restrictions of T to H and H', respectively. Let $\Gamma \subseteq H$ be as before. Now, by subdividing T if necessary, assume Γ satisfies both a) and b). Then $U(S'^{(1)})$ is obtained from $U((\partial S')^{(1)})$ by adding a handle. By adding the same handles to $U(S^{(1)})$, we get $U(T^{(1)})$. Now, the handlebody $U(S^{(1)})$ is obtained by adding handles to $U(\Gamma)$. We can summarize this as

$$U(\Gamma) \rightsquigarrow U(S^{(1)}) \rightsquigarrow U(T^{(1)}) = H(T)$$

where $H(T)$ is a handlebody in the Heegaard splitting of $Y = H(T) \cup H'(T)$ associated with the triangulation T and the arrows represent adding 1-handles. Thus, the original Heegaard splitting

$Y = H \cup H'$ is stably equivalent to the Heegaard splitting of $U(T)$ and its complementary handle body, and this is indeed a Heegaard splitting associated to a triangulation.

Finally, consider any two Heegaard splittings τ_1 and τ_2. By our second claim, τ_1 is stably equivalent to some Heegaard splitting σ_1 associated to a triangulation. Similarly, τ_2 is stably equivalent to some Heegaard splitting σ_2 associated to a triangulation. Furthermore, as σ_1 and σ_2 are associated to triangulations, they are stably equivalent, by our first claim. Hence,

$$\tau_1 \sim \sigma_1 \sim \sigma_2 \sim \tau_2$$

so any two Heegaard splittings are stably equivalent, as desired. □

3. Heegaard Diagrams

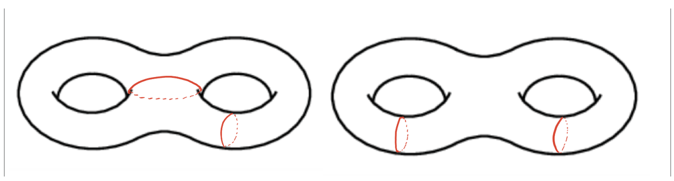
Now, to be able to turn Heegaard splittings into a chain complex, we will need more structure. Our first step is to add circles to the surface of the handlebodies.

Definition 3.1. A *set of attaching circles* $\gamma = (\gamma_1, \ldots, \gamma_g)$ for a handlebody H is a collection of closed embedded curves in $\Sigma_g = \partial H$ with the following properties:

- The curves γ_i are disjoint from each other.
- $\Sigma_g - \gamma_1 - \cdots - \gamma_g$ is connected
- The curves $\gamma_1, \ldots, \gamma_g$ bound disjoint embedded disks in H.

Let's see some examples of attaching circles.

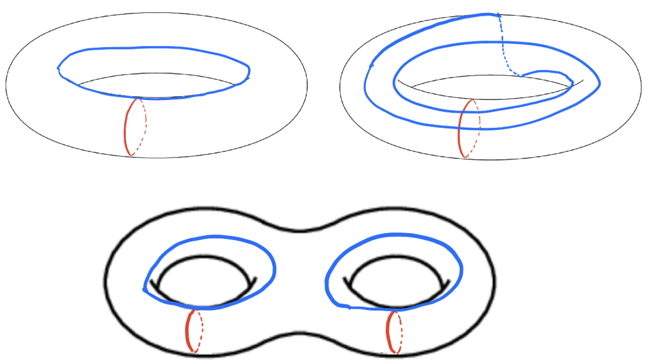
Example 3.2. Two possible configurations of attaching circles in Σ_1.



Now, more precisely, a Heegaard diagram is just a Heegaard splitting after adding attaching circles.

Definition 3.3. Let $Y = H \cup H'$ be a Heegaard splitting. A *Heegaard diagram* is a triple $\mathcal{H} = (\Sigma_g, \alpha, \beta)$, where , $\Sigma_g = \partial H$ and $\alpha = (\alpha_1, \ldots, \alpha_g)$ and $\beta = (\beta_1, \ldots, \beta_g)$ are sets of attaching circles for H and H', respectively.

Example 3.4. Some Heegaard diagrams.



For some technical details, we will also need to consider Heegaard diagrams with a basepoint.

Definition 3.5. A *Heegaard diagram with a basepoint* is a quadruple $\mathcal{H} = (\Sigma_g, \alpha, \beta, z)$, where $(\Sigma_g, \alpha, \beta)$ is a Heegaard diagram and $z \in \Sigma_g - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$.

From now on, we will use Heegaard diagrams and Heegaard diagrams with a basepoint interchangeably depending on whether we need to consider the basepoint or not.

4. Symmetric Products

Now, have associated Y with a Heegaard diagram. The next step is to associate Y to a configuration space, which we will later use to construct the chain complex.

Definition 4.1. Let given a space X, we define $\mathrm{Sym}^g(X) = X \times \cdots \times X/\mathfrak{S}_g$, where there are g copies of X and we are quotienting by the action of the symmetric group $\mathfrak{S}_g$.

Now, let us consider this in the context in which X is an orientable surface Σ_g. It turns out that "by a miracle called the Fundamental Theorem of Algebra," as said by Ozsváth, $\mathrm{Sym}^g(\Sigma_g)$ is a smooth manifold with a complex structure it inherits from Σ_g. Now, consider a Heegaard diagram $\mathcal{H} = (\Sigma_g, \alpha, \beta)$. Let us look at the subspaces of $\mathrm{Sym}^g(\Sigma_g)$ induced by the circles α and β.

Definition 4.2. Let $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g \in \mathrm{Sym}^g(\Sigma_g)$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g \in \mathrm{Sym}^g(\Sigma_g)$ be the subspaces of $\mathrm{Sym}^g(\Sigma_g)$ induced by the circles α and β. We call these subspaces *real tori*.

Note that, as manifolds, $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ have half the dimension of $\mathrm{Sym}^g(\Sigma_g)$. Indeed, $\dim \mathbb{T}_\alpha = \dim \mathbb{T}_\beta = g$, while $\dim \mathrm{Sym}^g(\Sigma_g) = 2g$. Now, when we have a basepoint z, we consider another subspace of of $\mathrm{Sym}^g(\Sigma_g)$. Let V_z be the subspace of $\mathrm{Sym}^g(\Sigma_g)$ induced by the basepoint z consisting of all the g-tuples of points in $\mathrm{Sym}^g(\Sigma_g)$ with z in at least one of their coordinates. A bit informally, we can say that

$$V_z = \{z\} \times \ \mathrm{Sym}^g(\Sigma_g) \tag{4.1}$$

Note that V_z has codimension 2. Now, let us focus on the subspace $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Assuming that the α_i and β_i intersect transversally (that is, that they only intersect at points and not in whole segments), then $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ consists of the intersections points. In other words, it has dimension 0.

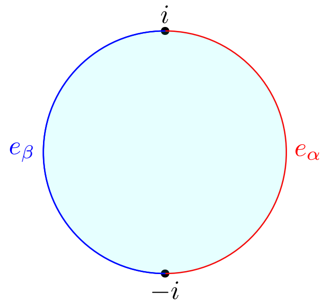
5. The Floer Chain Complex

Now, we are ready to begin our path towards the chain complex we want to relate to Y. Let $\mathbb{F}$ denote the field of two elements. That is, $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$. We begin with some technical definitions.

Definition 5.1. Given a Heegaard diagram $\mathcal{H} = (\Sigma_g, \alpha, \beta)$ for a 3-manifold Y, let $\widehat{CF}(\mathcal{H})$ denote the free Abelian group generated over $\mathbb{F}$ by the points $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

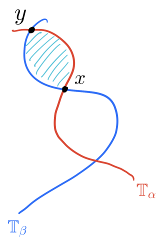
Definition 5.2. Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbb{D}$ be the unit disk in $\mathbb{C}$. A *Whitney disk* from x to y is a continuous map $\varphi : \mathbb{D} \to \mathrm{Sym}^g(\Sigma_g)$ with the following conditions:

- $\varphi(-i) = x$
- $\varphi(i) = y$
- $\varphi(e_\alpha) \subseteq \mathbb{T}_\alpha$
- $\varphi(e_\beta) \subseteq \mathbb{T}_\beta$



Whitney disk

Now, an embedding of a Whitney disk in $\mathrm{Sym}^g(\Sigma_g)$ looks roughly as follows:



Definition 5.3. Let $\pi_2(x, y)$ be the homotopy classes of Whitney disks.

Definition 5.4. Let $\mathcal{M}(\varphi)$ be the moduli space of the holomorphic representatives of φ.

It is known that $\mathcal{M}(\varphi)$ is a smooth manifold.

Definition 5.5. Let $\mu(\varphi)$ be the expected dimension of φ.

Definition 5.6. Let $n_z(\varphi) = \#(\varphi(\mathbb{D}) \cap V_z)$.

Note that there is an $\mathbb{R}$-action on $\mathcal{M}(\varphi)$ consisting on the complex isomorphism that fix $\pm i$. If $\mu(\varphi) = 1$, then

$$\widehat{\mathcal{M}}(\varphi) = \mathcal{M}(\varphi)/\mathbb{R} \tag{5.1}$$

is going to be 0 dimensional.

Definition 5.7. Let $\partial : \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H})$ be the map defined by

$$x \mapsto \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu_\varphi = 1 \\ n_z(\varphi) = 0}} \#\widehat{\mathcal{M}}(\varphi) y \tag{5.2}$$

The reason we defined such a ∂ is because of the following result of [**5**]:

Theorem 5.8. *$(\widehat{CF}(\mathcal{H}), \partial)$ is a chain complex.*

This is our desired chain complex.

6. Heegaard Floer Homology

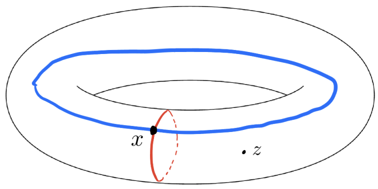
Finally, we are ready to define the Heegaard Floer homology for a 3-manifold Y.

Definition 6.1. The *Floer homology groups* $\widehat{HF}(Y)$ are the homology groups of the chain complex $(\widehat{CF}(\mathcal{H}), \partial)$. That is,

$$\widehat{HF}(Y) := H_*(\widehat{CF}(\mathcal{H})) \tag{6.1}$$

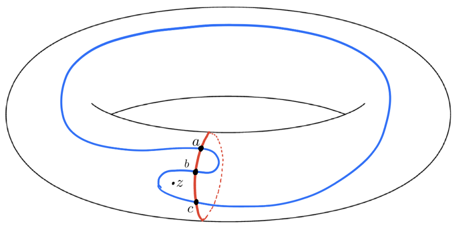
As an example, let us compute $\widehat{HF}(S^3)$ in two different ways using the Heegaard splitting $S^3 = H^1 \times H^1$. Let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Example 6.2. First, let us compute $\widehat{HF}(S^3)$ using the following Heegaard diagram.



Here, $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ consist of a single element, so $\widehat{CF}(\mathcal{H}) \cong \mathbb{F}$ and $\partial = 0$. Hence, $\widehat{HF}(S^3) \cong \mathbb{F}$.

Example 6.3. Now, let us use the following Heegaard diagram.



This example is a bit more interesting. Here, $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \{a, b, c\}$, so $\widehat{CF}(\mathcal{H}) \cong \mathbb{F}^3$. Now, let us compute ∂a, ∂b, and ∂c. By looking at the Whitney disks, note that $\partial a = b$. However, the Whitney disk involving b and c contains the basepoint z, so we get $\partial b = \partial c = 0$. Hence,

$$\widehat{HF}(S^3) \cong \langle b, c \rangle / \langle c \rangle \cong \mathbb{F} \tag{6.2}$$

which agrees with what we got before.

References

[1] P. Ozsváth and Z. Szabó. *An introduction to Heegaard Floer homology.*
[2] N. Saveliev. *Lectures on the topology of 3-manifolds: an introduction to the Casson invariant.*
[3] J. Greene. *Heegaard Floer homology.*
[4] B. Liu. *Heegaard Floer homology of L−space links with two components.*
[5] P. Ozsváth and Z. Szabó. *Holomorphic disks and topological invariants for closed three-manifolds.*
[6] J. Hom. *Heegaard Floer homology, Lectures 1-4.*

Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139